On Continuous Ambiguities in Model-Independent Partial Wave Analysis - II.

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Abstract

A problem of density matrix determination in terms of the given angular distribution of decay products is considered.

In the first part of this paper (Section III) a quantum system, described by density matrix, has been considered. In this the density matrix was diagonal with respect to orbital quantum numbers and the state was mixed only partially. Now we will consider the process $X \to 0 + 0$, where initial state is described by the density matrix ρ, for which we do not assume any other properties except hermitity and positive definiteness.

The angular distribution is given by the expression

$$I(\vec{n}) = \sum_{\alpha\beta} \rho_{\alpha\beta} Y^*_\alpha(\vec{n}) Y_\beta(\vec{n}). \tag{1}$$

Let us multiply both sides of this expression by $Y_\gamma(\vec{n})$ and integrate this over the sphere

$$t_\gamma = \sum_{\alpha\beta} \rho_{\alpha\beta} D_{\alpha\beta\gamma}, \tag{2}$$

where $t_\gamma = \int d^2n \; IY_\gamma =< Y_\gamma >_I$ are moments of distribution,

$$D_{\alpha\beta\gamma} = \int d^2n \; Y^*_\alpha Y_\beta Y_\gamma.$$

The integrals $D_{\alpha\beta\gamma}$ are expressed in terms of Clebsch-Gordan coefficients (see [1]§107):

$$D_{l_1 m_1, l_2 m_2, l_3 m_3} = (-1)^{m_3} \sqrt{\frac{(2l_1+1)(2l_3+1)}{4\pi(2l_2+1)}} \; C^{l_2 \; -m_2}_{l_1 \; -m_1, l_3 m_3} \; C^{l_2 0}_{l_1 0, l_3 0} \; .$$

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We use spherical functions [2], different from those of [1] by a phase factor: $Y^{[1]}_{lm} = i^l Y^{[2]}_{lm}$. Summation in (2) extends for the following values of indices:

$$m_2 = m_1 - m_3, \quad l_2 = |l_1 - l_3| \; ... \; l_1 + l_3 \;\; \text{step 2},$$

$$-\min\{l_1, l_2 - m_3\} \leq m_1 \leq \min\{l_1, l_2 + m_3\}.$$

Let us represent the density matrix $\rho_{\alpha\beta}$ via eigenvalues λ_i and eigenvectors Ψ^i_α

$$\rho_{\alpha\beta} = \sum_i \lambda_i \Psi^{i*}_\alpha \Psi^i_\beta.$$

The Ψ^i_α form unitary matrix

$$\sum_i \Psi^{i*}_\alpha \Psi^i_\beta = \delta_{\alpha\beta} \; .$$

The positivity of ρ implies that all eigenvalues are positive: $\lambda_i > 0$.
Let us rewrite (2)

$$\begin{aligned} t_\gamma &= \sum_i M_{\gamma i} \lambda_i, \qquad (3) \\ M_{\gamma i} &= \sum_{\alpha\beta} \Psi^{i*}_\alpha \Psi^i_\beta D_{\alpha\beta\gamma}. \end{aligned}$$

At $\gamma = 0$ from (3) a normalization condition follows:

$$\sum_i \lambda_i = \int d^2 n \; I = 1, \;\; \text{the distribution is normalized to unity.}$$

At the given Ψ^i_α relation (3) is a system of linear equations for λ_i. A matrix of this system is non-degenerate, except rare special values of Ψ^i_α.

> Particularly, the matrix $M_{\gamma i}$ degenerates when $\Psi^i_\alpha = \delta_{i\alpha}$. The question whether the matrix $M_{\gamma i}$ is non-degenerate reduces to clarification of linear independence of a set of functions $|\Psi_i(\vec{n})|^2$ for the given complete set of functions $\Psi_i(\vec{n})$.

Therefore in our approach the free parameters are the unitary matrices Ψ^i_α, the values λ_i are expressed via Ψ^i_α from system (3), the inequalities $\lambda_i(\Psi) > 0$ select a region of possible values for Ψ.

Notion.

The presence of continuous ambiguity in the solution of this problem is most obvious in $\vec{n}$-representation of operators. The density matrix is represented by the kernel of integral operator

$$(\rho\Psi)(\vec{n}) = \int d^2 n' \rho(\vec{n}, \vec{n}') \Psi(\vec{n}').$$

The distribution is equal to the diagonal elements

$$I(\vec{n}) = \rho(\vec{n}, \vec{n}) \tag{4}$$

Hence, the problem is reduced to the reconstruction of hermitean positive matrix from the given diagonal. Let ρ be a solution of this problem. The replacement $\rho \to \rho + \Delta\rho$, where $\Delta\rho$ is arbitrary hermitean matrix with zero diagonal elements, conserves (4) and hermitity of ρ. If the matrix elements of $\Delta\rho$ are sufficiently small, then the positivity of ρ is also conserved. Therefore, the solution of the problem is not unique.

In $\vec{n}$-representation the analysis of ρ positivity and the determination of eigenvectors are complicated. The approach proposed above seems more convenient.

Scheme of analysis

When describing quantum systems via density matrix one suggests that the system is located in pure states Ψ^i_α with different probabilities λ_i. The representation of eigenvectors of density matrix on Argand plots is an ultimate goal of partial wave analysis. The input data are the moments t_γ. We propose to obtain the region of solution using Monte-Carlo technique. The scheme of analysis is the following:

1. A random unitary matrix Ψ^i_α is generated. We use the generation of the columns of this matrix as a set of independent isotropically distributed random complex vectors with the subsequent Hilbert-Schmidt orthogonalization.

   An isotropic (Gaussian) distribution of n-dimensional vector can be obtained by generating its components as independent random values with Gaussian distribution:

   $$dp = e^{-{x_1}^2} dx_1 \ ... \ e^{-{x_n}^2} dx_n = e^{-r^2} dV.$$

   The unitary matrices generated by this method are uniformly distributed over the unitary group invariant volume[1].

2. The matrix $M_{\gamma i}$ is obtained, system (3) is solved for λ_i.

3. The solution is accepted, if all $\lambda_i > 0$, and is discarded otherwise.

4. The eigenvalues λ_i are displayed on intensity plots, the eigenvectors Ψ^i_α – on Argand plots.

[1] The unitary transformation transfers one orthonormal basis to another. The invariant measure on unitary group can be defined as a product of spherical volume for one vector from the basis and the measure on subgroup leaving this vector invariant (little subgroup). Using this recurrent definition and isotropy of generated vectors, one can easily prove the above statement by induction.

Remarks.

1) Phase rotation of eigenvectors $\Psi^i_\alpha \to e^{i\varphi_i}\Psi^i_\alpha$ does not change the density matrix. One can generally choose the phases of eigenvectors in such a way that the component Ψ^i_0 in each eigenvector will be a real positive number (phase is measured from S-wave). Those unitary $N\times N$ matrices are described by N^2-N independent real parameters, which being unified with N eigenvalues λ_i give N^2 degrees of freedom, contained in an arbitrary hermitean matrix $\rho_{\alpha\beta}$.

2) Change of numeration (permutation) of eigenvalues and eigenvectors conserves the density matrix. The region of solutions is symmetrical with respect to these permutations. Thus projections of the region onto Argand plots of each eigenvector coincide. One can order the eigenvalues: $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N$ and respectively permute the eigenvectors. Such ordering might lead to breaks of continuity in solutions. The breaks appear when a collision of eigenvalues occurs on a smooth solution (fig.1). The ordering transfers the continuous path ab to the non-continuous path ab'. One should take into account this fact when selecting the solutions: not only smooth paths inside ambiguity region are allowed, but also those paths, which can be smoothly continued in the points with coincident eigenvalues after the permutation of the correspondent eigenvectors.

> In the vicinity of the point A the eigenvectors Ψ_1 and Ψ_2 cannot be close because they are orthogonal. The points A and A' cannot coincide on all Argand plots. Hence the break of curves on Argand plots actually occurs in transition through the point A along the path ab'. One can connect the points A and A' by continuous path, along which the density matrix is constant. At $\lambda_1 = \lambda_2$ any linear combinations of Ψ_1 and Ψ_2 are also eigenvectors. The Ψ_1 and Ψ_2 can be permuted with the aid of unitary transformation, not touching others Ψ_i (this can be developed continuously, because $U(2)$ group is connective). The rotation AA' is developed at constant $\lambda_1 = \lambda_2$. Hence, though the transition $aAA'b'$ is continuous, it cannot have analytical dependence on energy.)

3) Let first N moments of distribution be measured in experiment ($l \leq L_{max},\ N = (L_{max}+1)^2$). Let the unitary matrices Ψ have sizes $N \times N$. We will solve N first equations of (3) for N unknowns λ_i. The density matrices obtained in solution give the distributions, for which N first moments exactly coincide with measured ones. Other moments $t_\gamma,\ \gamma > N$ can be obtained by substitution of values Ψ^i_α, λ_i into (3). Generally these moments do not vanish. One can show from the properties of the coefficients $D_{\alpha\beta\gamma}$ that there is a finite number of non-zero moments: $L_{max} < l \leq 2L_{max}$.

The partial wave analysis practice is to treat higher (non-measured) moments as zeros with the experimental precision. One can impose conditions on these moments

$$|t_{lm}| < \delta t, \qquad L_{max} < l \leq 2L_{max},$$

where δt is a statistical error of the moment.

$$|\delta t_\gamma|^2 = \frac{DY_\gamma}{N_{\rm evts}},$$

$N_{\rm evts}$ is the total number of events, on which the moment is measured

$$t_\gamma = \frac{1}{N_{\rm evts}} \sum_{i=1}^{N_{\rm evts}} Y_\gamma(\vec{n}_i),$$

DY_γ is the dispersion of random value $Y_\gamma(\vec{n})$,

$$DY_\gamma = < |Y_\gamma|^2 > - | < Y_\gamma > |^2 = \int d^2n \; I|Y_\gamma|^2 \; - |t_\gamma|^2,$$

$$|\delta t_\gamma|^2 \leq \frac{1}{N_{\rm evts}} \int d^2n \; I|Y_\gamma|^2.$$

Averaging both sides over the quantum number m, we have

$$\frac{1}{2l+1} \sum_m |\delta t_{lm}|^2 \leq \frac{1}{4\pi N_{\rm evts}}, \quad \text{i.e. } \overline{|\delta t_{lm}|} \leq \frac{1}{\sqrt{4\pi N_{\rm evts}}} = \delta t.$$

When $|t_\gamma| > \delta t$, the deviation of t_γ from zero is statistically considerable.

We note, that our estimate for δt is l independent, all moments are measured with equal precision. Actually one can reliably measure a great number of moments (limitations are imposed only by finite angular distribution of the equipment: $l < 2\pi/\Delta\theta_{\rm resol}$).

The contributions of the moments with high l in distribution $I(\vec{n})$ are fast oscillating functions. To detect l-th harmonics one should fill 2D histograms, containing at least l^2 bins. Due to limited statistics a few events will be placed in one bin. Hence, the presence of high harmonics can not be detected in angular distribution. Nevertheless, the moments of these harmonics can be precisely measured.

Example. Let us consider the distribution

$$I(\vec{n}) = (1-a)|Y_0|^2 + a|Y_{10}|^2 = \frac{1}{4\pi}(1 - a + 3a\cos^2\theta). \tag{5}$$

The moments are

$$t_0 = \frac{1}{\sqrt{4\pi}}, \quad t_{20} = \frac{a}{\sqrt{5\pi}}, \quad \text{others } t_\gamma = 0. \tag{6}$$

Let first 9 moments t_γ be measured in an experiment (S, P and D-moments)[2]. The density matrix is described by 81 real parameters, 72 of which describe the unitary matrix Ψ ($\Psi_0^i \in R_+$), 9 eigenvalues λ_i can be obtained from system (3).

Figures 2 and 3 show the result of analysis for the value $a = 1/6$. The number of random unitary matrices was 100000. In 1646 cases the solution of system (3) satisfied the condition $\lambda_i > 0$. For 54 points the additional condition $|t_{lm}| < 10^{-2},\ l \geq 3$ held true.

The solutions Ψ fill a region in the group $U(9)$. In projection onto Argand plots this region maps into a set of related points. The density of points on figures estimates a "thickness" of the stratum, projected into the same area on the plot. A circle on the figures denotes the trivial solution

$$\begin{aligned} \lambda_1 &= 5/6 \quad \Psi_0^1 = 1, \quad \text{others } \Psi_\alpha^1 = 0 \quad \text{(S-wave)} \\ \lambda_2 &= 1/6 \quad \Psi_{10}^2 = 1, \quad \text{others } \Psi_\alpha^2 = 0 \quad \text{(P-wave)} \\ \lambda_i &= 0,\ i = 3..9 \end{aligned}$$

A few points fall in the vicinity of the trivial solution, most part of solutions is located in the region of close eigenvalues $\lambda_i = 0...0.2$.

The points, for which the additional condition $|t_{lm}| < 10^{-2},\ l \geq 3$ is satisfied, cover the same areas on the figures. A position of these solutions is analogous to the position of thin layer in a sphere: outer spherical layer occupies a small volume in a sphere, but covers the same area in projection on a plane.

Limiting cases.

$a \to 0$ Isotropic distribution ($a \to 0$) is a singular case for the given scheme of analysis. When $t_\gamma = \delta_{\gamma 0}/\sqrt{4\pi}$, a set of equal eigenvalues $\lambda_i = 1/N$ is a solution of (3) for all Ψ. In this the density matrix is proportional to unit matrix: $\rho_{\alpha\beta} = \delta_{\alpha\beta}/N$, the isotropy of distribution (1) follows from the property $\sum_{m=-l}^{l} |Y_{lm}(\vec{n})|^2 = (2l+1)/4\pi$. The solution $\lambda_i = 1/N$ has no definite limit at $N \to \infty$ and it is not of physical value. Also there are other solutions, e.g. pure S-wave state or

$$\Psi = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & U(3) & 0 & \dots \\ 0 & 0 & U(5) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \quad , \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 \cdot 1_{3\times 3} & 0 & \dots \\ 0 & 0 & \lambda_3 \cdot 1_{5\times 5} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \quad ,$$

the matrix Ψ is diagonal with respect to l (mixes m only), correspondent λ_{lm} are independent of m. System (3) has these additional solutions only if the matrix $M_{\gamma i}$ is degenerate. The values of parameters Ψ, for which $M_{\gamma i}$ degenerates, form in unitary group a set of zero measure. When generating matrices Ψ uniformly distributed in $U(N)$ volume these additional solutions will not be revealed.

[2]The distribution can differ from (5) by the contributions of higher moments $l \geq 3$.

Transition to the limit $a \to 0$ is shown on fig.4. The distribution of λ_i contracts to the point $\lambda = 1/9$ ($N = 9$), the points on Argand plots fill unit circles.

The presence of such singularities indicates that the whole set of solutions of system (3) possesses a complicated topological structure. This structure can be inferred from the following examples (fig.5):

$$\text{a)} \quad \begin{pmatrix} 1 & 1 \\ x & y \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{y-x} \begin{pmatrix} y \\ -x \end{pmatrix}.$$

At $x = y = 0$ the solutions of the system (with positive λ_i) form a segment ($0 < \lambda_1 < 1,\ \lambda_2 = 1 - \lambda_1$). This set has greater dimension ($d = 1$, a line) than the set of solutions at fixed $x \neq y$ ($d = 0$, a point), but less dimension than the set of solutions for all x, y ($d = 2$, a surface). Singular solutions have zero measure in the set of all solutions (on the hyperbolic paraboloid, fig.5a).

$$\text{b)} \quad \begin{pmatrix} 1 & 1 & 1 \\ x & -x & 0 \\ y & 0 & -y \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

At fixed $x \neq 0, y \neq 0$ the solution is a point $\lambda_i = 1/3$. When $x = 0$ or $y = 0$, the dimension of the set of solutions is increased by 1, when $x = y = 0$ – by 2. The set of singular solutions has the same dimension as the set of all non-singular ones ($d = 2$).

In uniform random generation of parameters (x, y) the singular solutions will not be found. This is essential only for singularities of the type b).

Limits of a. Function (5) is positive on the sphere at $-1/2 < a < 1$. For these values of a the problem considered has solutions.

Remarks.

1) At negative a the point $\{\lambda_1 = 1-a,\ \Psi_1 = \text{S-wave};\ \lambda_2 = a,\ \Psi_2 = \text{P-wave}\}$ is no longer a positively defined solution of the problem.

2) Non-negativity of the distribution is necessary and sufficient condition for positively semi-defined density matrix existence, if sizes of matrices are not bonded ($N \to \infty$). The necessity is obvious, the sufficience follows from the fact that any distribution $I(\vec{n}) \geq 0$ can be described by a pure state (see first part of this paper, Section I), i.e. the density matrix with single non-zero eigenvalue

$$\lambda_1 = 1,\ \lambda_i = 0, i > 1,\ \Psi_1(\vec{n}) = \sqrt{I(\vec{n})} e^{i\varphi(\vec{n})}. \tag{7}$$

3) At finite N this condition is neither necessary nor sufficient. We study a class of distributions which can differ by harmonics with numbers greater than N. Even if $I(\vec{n}) < 0$ in some point for distribution considered, positive functions can exist in the same class. On the other hand, the pure states of form

Table 1: Number of positively defined solutions for 50000 random matrices with random distribution in $U(9)$

a	-1/4	-1/8	-1/16	0	1/16	1/8	1/6	1/4
$N_{\rm sol}$	9	2192	14689	50000	14969	2770	823	75

(7) have in general an infinite number of harmonics and are not contained in the considered finite dimensional class Ψ^i_α. One can show that a set of a values, for which the problem has solutions in the finite dimensional class, is connective, i.e. it is a segment $a_- \le a \le a_+$, with $a_+ \ge 1$ for distribution (5). We will not determine exact values $a_\pm$.

When a tends to boundary points, the volume occupied by solutions in $U(N)$, tends to zero fast. The behaviour of this volume could be inferred from the data given in Table 1.

At large $|a|$ and at a greater number of moments involved in analysis as well, the Monte-Carlo based approach presented here is ineffective (most part of solutions is discarded in positivity test λ_i). One might use more advanced methods for solution of inequalities system $\lambda_i(\Psi) > 0$.

Discussion

The pure states are particular cases of mixed states, hence the ambiguity region for mixed states is wider than for pure ones. Even in the class of solutions, containing a finite number of harmonics, continuous transformations of density matrix are available, which do not change the distribution. One can fix the continuous ambiguity of solutions only imposing model restrictions on the form of density matrix. The illustration of this statement can be found in [3]. This work presents a formalism used in partial wave analysis of the reactions $\pi p \to (3\pi)p,\ Kp \to (K\pi\pi)p$. The process is represented in a form of decay sequence[3]

$$\text{meson } + p \to p + \text{ meson } (J^P);$$

$$\text{meson } (J^P) \to \text{ meson } + \text{ di-meson };$$

$$\text{di-meson } \to \text{ 2 mesons }.$$

Then assumptions for the amplitude (eigenvectors of density matrix) are made, of which the most important are:

[3]It is assumed that the process amplitude has a general form and depends on all quantum numbers and kinematic variables. In this step the representation of the process as decay sequence is introduced for convenience and do not impose any restrictions on the form of amplitude.

1. The amplitude dependence on kinematic variables, describing the decay of di-meson, is introduced as a product of Breit-Wigner function and barrier factors.

2. The amplitude of the process is represented as a product of J^P state production amplitude and its decay amplitude. This assumption is exact only if a single state J^P is present.

As stated in the work, namely these assumptions fix continuous ambiguity of partial wave analysis (1–partially, 2–completely).

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References

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Figure captions

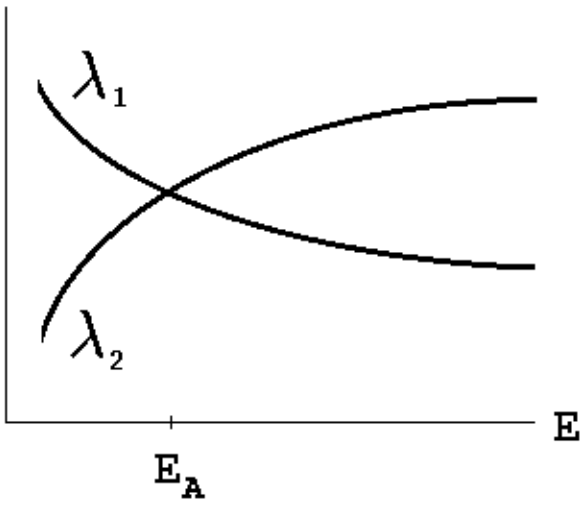
Fig.1. In point A the eigenvalues λ_1 and λ_2 coincide. The continuous path ab transfers after ordering of eigenvalues into the path ab', lying in the region $\lambda_1 > \lambda_2$ (hatched on fig.b). The eigenvectors Ψ_1 and Ψ_2 can not coincide (the case (c) is impossible). In the point A the break of curves on Argand plot occurs (fig.d).

Fig.2. Ambiguity region: eigenvalues. A circle denotes the trivial solution (mixture of S and P-waves). One arbitrarily selected solution is denoted by a star.

Fig.3. Ambiguity region: eigenvectors. a) Ψ_1; b) Ψ_2.

Fig.4. Ambiguity region: $a \to 0$.

Fig.5. Parametric dependence of linear systems solutions (see text).



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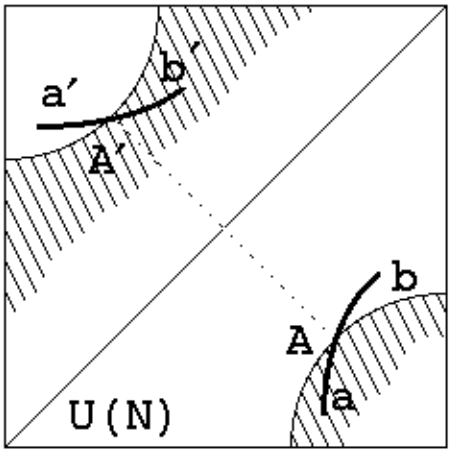

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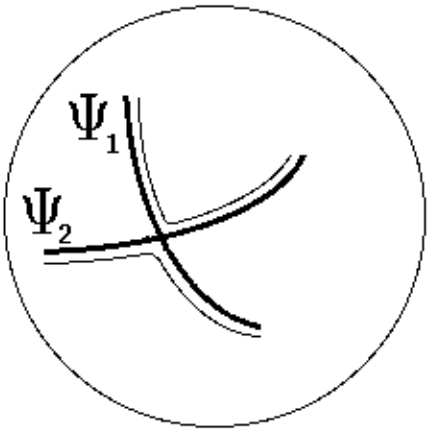

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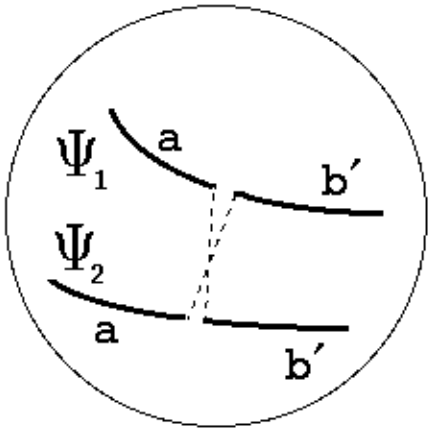
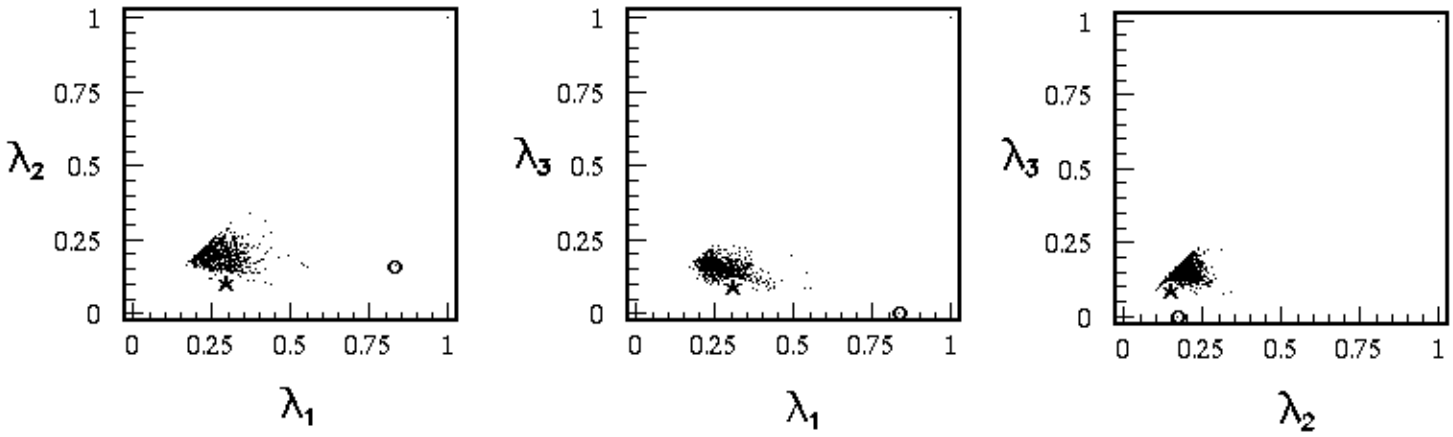


d)

fig.1


1
0.75
0.5
0.25
0
λ_2
0
0.25
0.5
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1
λ_1
λ_3
λ_1
λ_3
λ_2

fig.2

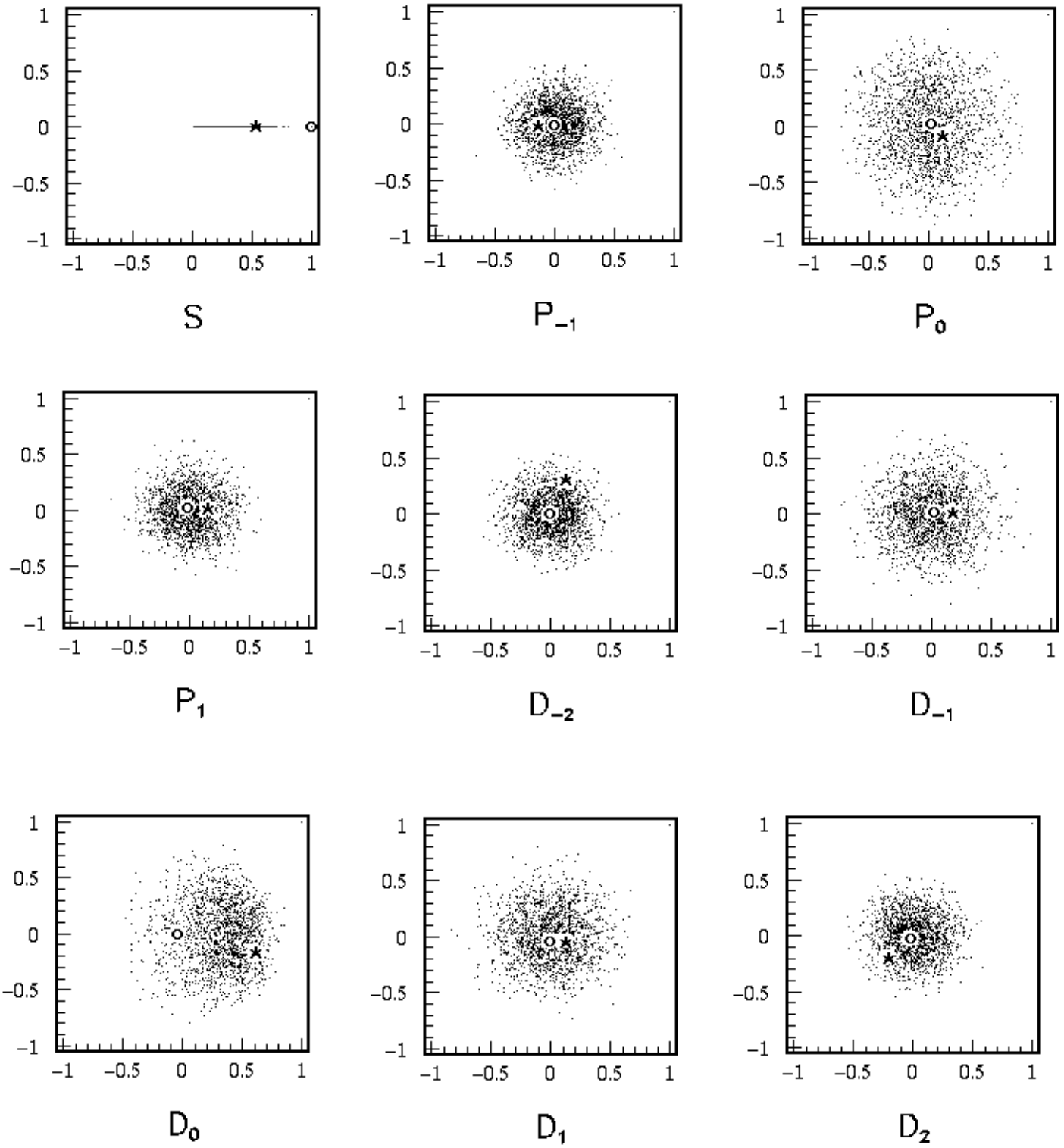


fig.3a

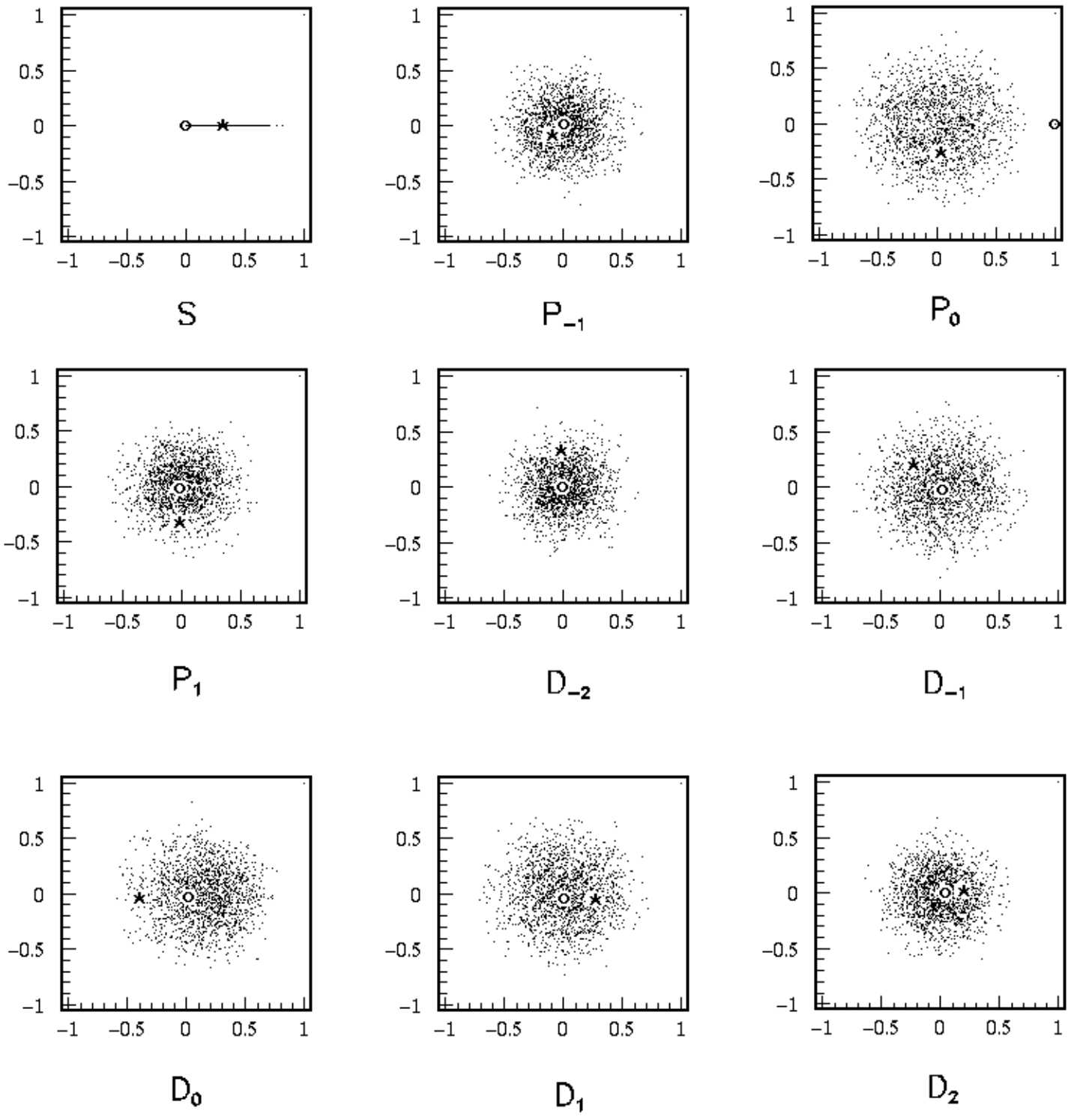


fig.3b

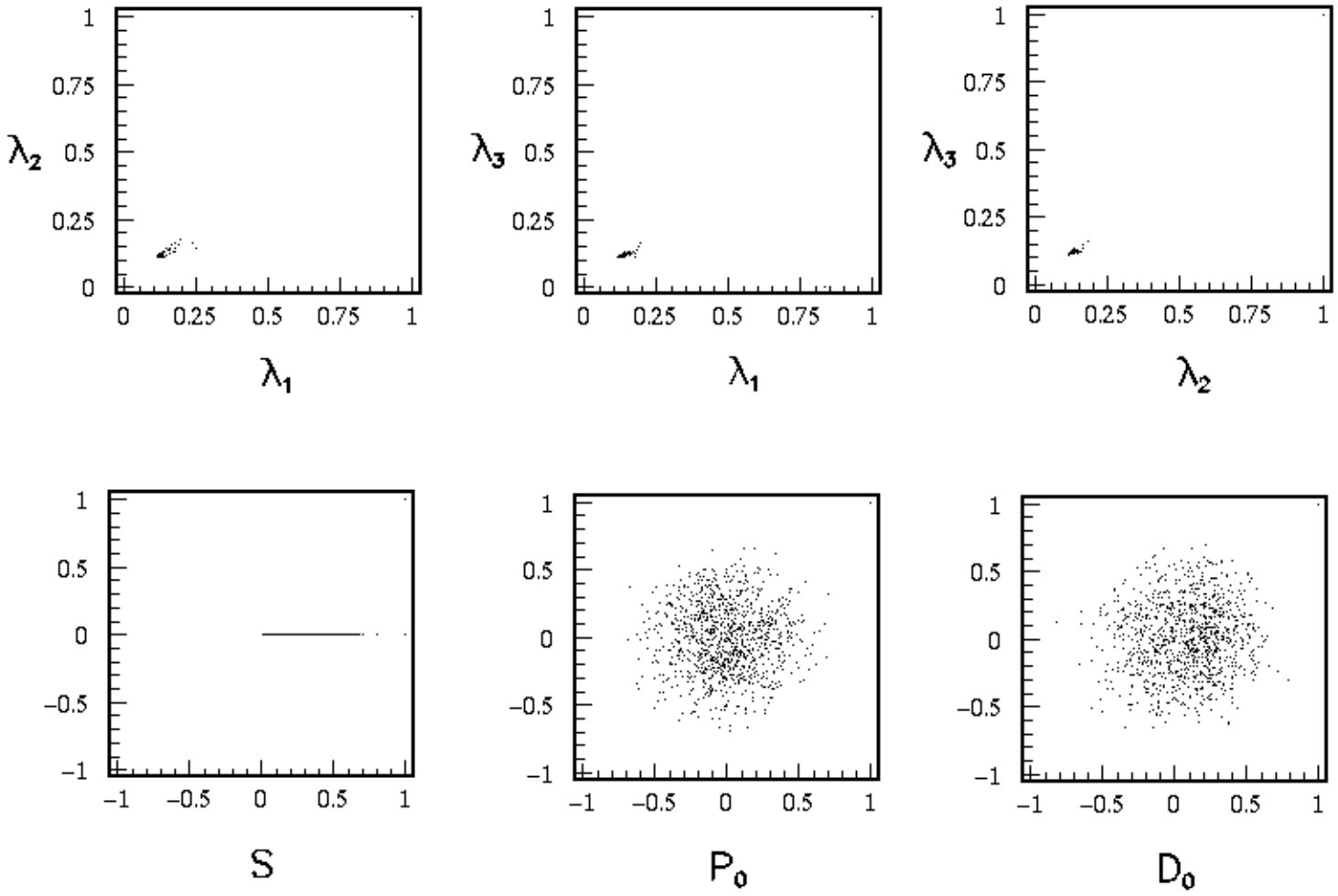
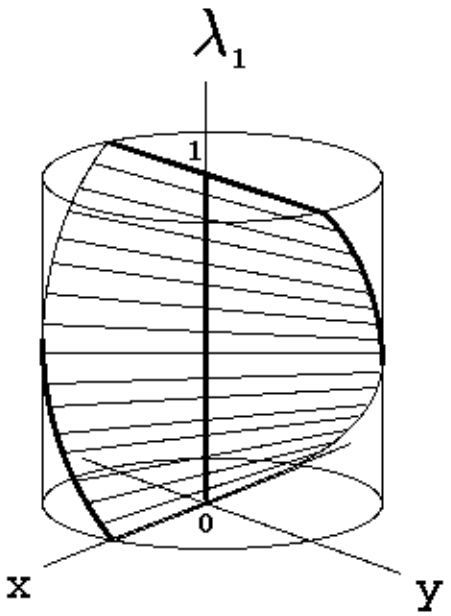
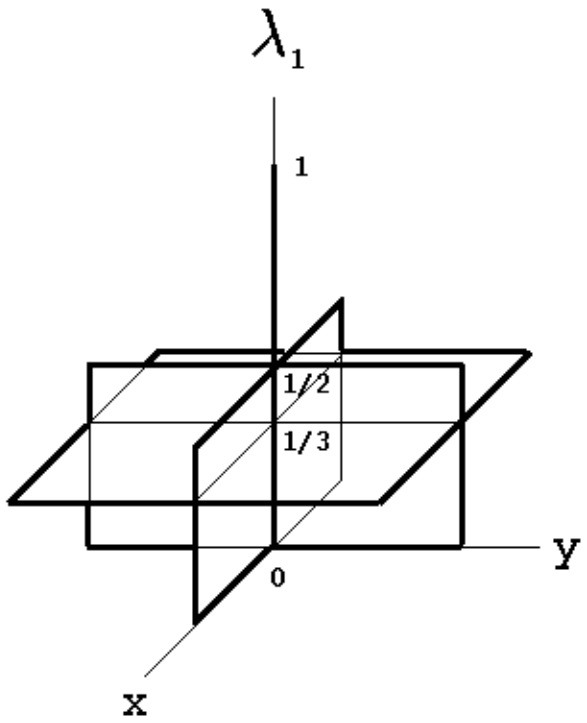


fig. 4



a)



b)

fig. 5